40018A – Discrete Maths and Logic Answers

(Slightly questionable “logic” but slightly more confident with discrete lol steffen is life <3 Ty to Gabriel for uploading answers for Q2)

1a) V = { {a}, {b}, {c} }, W = { {a}, b, c }

V ∩ W = { {a} }

V ∪ W = { {a}, {b}, {c}, b, c }

℘ W = { ∅, { {a} }, {b}, {c}, { {a}, b }, { {a}, c}, { b, c }, { {a}, b, c } }

V ∩ ℘ W = { {b}, {c} }

V △ W = { {b}, {c}, b, c }

1b) (literally the same 1b from the 2021 paper, just with ⟨d, c⟩, ⟨e, f⟩ instead of ⟨c, d⟩, ⟨f, e⟩)

A = { a, b, c, d, e, f }; ⟨a, b⟩, ⟨a, f⟩, ⟨c, d⟩, ⟨f, e⟩ ∈ R; R is reflexive, symmetric, transitive

R is reflexive -> ⟨a, a⟩, ⟨b, b⟩, ⟨c, c⟩, ⟨d, d⟩, ⟨e, e⟩, ⟨f, f⟩

R is symmetric -> ⟨b, a⟩, ⟨f, a⟩, ⟨d, c⟩, ⟨e, f⟩

R is transitive -> ⟨a, e⟩, ⟨e, a⟩, ⟨b, f⟩, ⟨f, b⟩, ⟨b, e⟩, ⟨e, b⟩

Diagram

Description automatically generated

1c) (There is a very similar question to this in the exercises, whose Steffen-approved answer I am using the structure of here, so always make sure you go through and review the exercises as there are often similar questions on exams!)

We need to show that R is reflexive, transitive, and anti-symmetric:

(⟨p, q⟩ R ⟨p, q⟩): There exists n = 1 and m = 1 such that both n × p = 1 × p = p and m × p = 1 × q = q.

(⟨p, q⟩ R ⟨r, s⟩ ∧ ⟨r, s⟩ R ⟨t, u⟩ **→** ⟨p, q⟩ R ⟨t, u⟩): if ⟨p, q⟩ R ⟨r, s⟩, there exists an a ∈ ℕ \ {0} such that a × p = r, and a b ∈ ℕ \ {0} such that b × q = s.

Similarly for ⟨r, s⟩ R ⟨t, u⟩, there exists a c ∈ ℕ \ {0} such that c × r = t, and a d ∈ ℕ \ {0} such that d × s = u.

But then, a × p = r and t = c × r = c × a × p, and similarly b × q = s and u = d × s = d × b × q.

Therefore, there exist values (c × a) ∈ ℕ \ {0} and (d × b) ∈ ℕ \ {0} such that t = (c × a) × p and u = (d × b) × q, and hence ⟨p, q⟩ R ⟨t, u⟩.

(⟨p, q⟩ R ⟨r, s⟩ ∧ ⟨r, s⟩ R ⟨p, q⟩ **→** ⟨p, q⟩ = ⟨r, s⟩): For ⟨p, q⟩ R ⟨r, s⟩, there exists an a ∈ ℕ \ {0} such that a × p = r and a b ∈ ℕ \ {0} such that b × q = s.

Similarly, for ⟨r, s⟩ R ⟨p, q⟩, there exists a c ∈ ℕ \ {0} such that c × r = p and a d ∈ ℕ \ {0} such that d × s = q.

But then a × p = r and c × r = p hence r = a × p = c × a × r. For (c × a) ∈ ℕ \ {0}, the only value of (c × a) for which this is valid is c × a = 1. Similarly, b × q = s and d × s = q hence s = b × q = d × b × s. For (d × b) ∈ ℕ \ {0}, the only value of (d × b) for which this is valid is d × b = 1.

Hence, c × a = d × b = 1, so r = p and s = q meaning ⟨p, q⟩ = ⟨r, s⟩.

We have now shown all 3 properties of a partial order to be true for R.

(N.B. There is likely some waffle which, in an exam setting, you could cut out and write shorter sentences to save precious time)

1d) |V| = 2 and |W| = 3

|℘ V| = 22 = 4, |℘ W| = 23 = 8

For every element in ℘ V, there are |℘ W| = 8 choices of images, hence there are |℘ W||℘ V| = 84 = 4096 possible functions.

For partial functions, we take the image set to be ℘ W ∪ {⊥}. The cardinality of the image set is now |℘ W ∪ {⊥}| = 23 + 1 = 9. There are now 9 choices of image for elements in ℘ V, so there are |℘ W ∪ {⊥}||℘ V| = 94 = 6561 partial functions from ℘ V to ℘ W.

1e) (Note there are many ways to do this question, this is just my preferred method)

From the lecture notes, page 45 example 6.4, a bijection h: ℕ **→** ℚ is constructed in the proof that ℚ is countable and hence, ℕ ≈ ℚ.

We now construct a function g: ℕ2 **→** ℚ2 and prove that this is a bijection between ℕ2 and ℚ2.

We define the function g like so:

g(⟨n1, n2⟩) = ⟨h(n1), h(n2)⟩

For a function to be surjective, for f: A **→** B, ∀b ∈ B . ∃a ∈ A . (f(a) = b). We know, from the result ℕ ≈ ℚ, that h is a bijection from ℕ to ℚ. The output of function g is a pair ⟨h(n1), h(n2)⟩. We know h is a bijection (and is therefore surjective), hence we know that h(n1) = q1 and h(n2) = q2 for arbitrary n1, n2 ∈ ℕ and q1, q2 ∈ ℚ. Hence g is also surjective.

For a function to be injective, for f: A **→** B, ∀a, a’ ∈ A . (f(a) = f(a’) → a = a’). We take arbitrary g(⟨n1, n2⟩) and g(⟨n1’, n2’⟩) and assume g(⟨n1, n2⟩) = g(⟨n1’, n2’⟩). The output values of each function call are determined by the function h, and since we know that h is bijective (and hence injective), if n1 = n1’, h(n1) = h(n1’), and similarly if n2 = n2’, h(n2) = h(n2’). Therefore, g is also injective.

We have shown g to be both surjective and injective, and hence g is a bijection. Therefore, we can determine that ℕ2 ≈ ℚ2 (as a bijection exists between ℕ2 and ℚ2).

We know from the lecture notes, page 42 example 5.33, that a bijection exists between ℕ and ℕ2, and hence ℕ ≈ ℕ2. We also know that ℕ2 ≈ ℚ2, hence by transitivity of ≈ we can therefore determine that ℕ ≈ ℚ2 (and ℚ2 ≈ ℕ by symmetry of ≈), and hence ℚ2 is countable (by Definition 6.1, page 45 in the lecture notes\*).

(\*Note Definition 6.1 (Countability) is as follows: A set A is countable if A is finite or A ≈ ℕ)

2aiA) (Super fun really cool question with approximately 463856 different interpretations, here’s what I did in the exam – see also Gabriel’s pdf which has another interpretation)

We define the atoms as such:

c = card is activated

m = there is money in the account

p = pin code is requested

d = payment is declined

¬(c ∧ m) **→** (¬p ∨ d)

Or

(c ∧ m) ∨ (¬p ∨ d)

Diagram

Description automatically generated2aiB) (Would be different for your formula if you had a different interpretation to me for 2aiA)

Sub-formulae:

¬(c ∧ m) **→** (¬p ∨ d)

¬(c ∧ m)

c ∧ m

¬p ∨ d

(Atoms – still sub-formulae): c, m, ¬p, d

2aiC) (Probably a bit dodgy but the logic is there. Basically copied it in the format it was introduced in the slides so hopefully it should provide some clarity? Maybe? Man, if you’re as confused as me right now just skip this question in the exam lol)

For the set of propositional atoms A = { c, m, p, d }, and the function *v*, an atomic evaluation function of A, the evaluation function |…|*v* assigns the truth value *true* (tt) or *false* (ff) to formulas as follows:

*v*(c) = tt, *v*(m) = tt, *v*(p) = tt or ff, *v*(d) = tt or ff

(Note: the |…|*v* notation is used when saying how conjunctions/disjunctions/etc of the atoms would evaluate under the function mapping *v* of atoms to true/false values given above - see use below in the justification of my answer)

The antecedent of the implication always evaluates to false under *v* (that is, |¬( c ∧ m)|*v* = ff), and by semantics of the implication operator, this means the formula will always evaluate to true (tt) regardless of the truth value of either p or d.

2aii) (can be done via two methods, 1. Using equivalences to change the formula unto CNF, or 2. Using a Computer Systems style approach as done here with truth tables and finding the maxterms)

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| P | Q | R | P **→**  ¬(Q ∨ R)) | R **→**  ¬P | (P **→**  ¬(Q ∨ R))) ∨ (R **→**  ¬P) | Maxterms |
| 0 | 0 | 0 | 1 | 1 | 1 |  |
| 0 | 0 | 1 | 1 | 1 | 1 |  |
| 0 | 1 | 0 | 1 | 1 | 1 |  |
| 0 | 1 | 1 | 1 | 1 | 1 |  |
| 1 | 0 | 0 | 1 | 1 | 1 |  |
| 1 | 0 | 1 | 0 | 0 | 0 | ¬P ∨ Q ∨ ¬R |
| 1 | 1 | 0 | 0 | 1 | 1 |  |
| 1 | 1 | 1 | 0 | 0 | 0 | ¬P ∨ ¬Q ∨ ¬R |

Formula in CNF:

(¬P ∨ Q ∨ ¬R) ∧ (¬P ∨ ¬Q ∨ ¬R)

2aiii) First assume that ⊧ φ ↔ ψ holds. So, in an arbitrary situation either φ and ψ are both true or both false. So, if we take the LHS of the disjunction we have φ ∧ ¬ ψ which evaluates to false, as the negation of ψ means we have ⊤ ∧ ⊥ or ⊥ ∧ ⊤. By a similar argument, the RHS of the disjunction evaluates to false. So, the formula is false by semantics of ∨. Since this is an arbitrary situation, we therefore have that the formula is always false, therefore unsatisfiable.

In the other direction, assuming (φ ∧ ¬ψ) ∨ (¬φ ∧ ψ) is unsatisfiable, this means the formula always evaluates to false. Therefore, we have that on the LHS, which evaluates to false, φ ∧ ¬ψ so we have both φ is false and ¬ψ is true. If ¬ψ was false then on the RHS we would have the conjunction evaluate to true which is impossible due to the assumption that the formula is unsatisfiable. Therefore, φ is false and ψ is false, or φ is true and ψ is true. This is the same assignment as in the other direction, hence ⊧ φ ↔ ψ 0 holds iff (φ ∧ ¬ψ) ∨ (¬φ ∧ ψ) is unsatisfiable.

2biAi) This says every circle that does not have an outgoing arrow, or every circle does not have an incoming arrow, so:

x = { 6, 3, 1 }

Using De Morgan’s this means every circle which does not have both and incoming and outgoing arrow, so only 6, 3, and 1 satisfy this condition.

2biAii) This says all black circles which have one or two outgoing arrows, so:

x = { 6 }

The other black circles have more than two outgoing arrows.

2biAiii) This is saying that there are at least two outgoing arrows that connect to white circles from x, so:

x = { 4, 2, 6 }

As 4, 2, and 6 have outgoing arrows that connect to at least two white circles each.

2biB) (Probably dodgy lol, haven’t had to do FOL in a while)

∀x (∀y (R(x, y) **→** F(y)) **→** F(x))

We go through the nodes of the diagram one by one to determine whether or not they are included in the interpretation of F or not.

x = 1: ∀y (R(1, y)) is always false (as 1 does not point to any other circles), hence the first implication always evaluates to true, therefore F(1) must be true.

x = 3 and x = 5 follow the same reasoning as above, so both F(3) and F(5) are also true.

x = 6: R(6, 3) and R(6, 5) are both true, and so are F(3) and F(5) (from above) hence the first implication evaluates to true. Hence, for the second to also evaluate to true, F(6) would also have to hold (by semantics of implies)..

x = 2: R(2, 1), R(2, 3) and R(2, 5) hold. By the same reasoning as above, F(2) must therefore hold. For R(2, 2) (which holds) this interpretation also holds, as F(y) = F(x) = ⊤.

x = 4: Same reasoning as above for R(4, 1), R(4, 5), and R(4, 4)

Therefore, the interpretation of F must be that F(x) is true for x = { 1, 2, 3, 4, 5, 6 }

2bii) Chart, treemap chart

Description automatically generated